Problem Set: The Solow–Swan Growth Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Unless otherwise stated, assume

$$0 < \alpha < 1, \quad 0 < s < 1, \quad n \geq 0, \quad g \geq 0, \quad 0 < \delta < 1, \qquad D \equiv (1+n)(1+g), \quad b \equiv D - (1-\delta) = n + g + ng + \delta.$$

The standard Solow approximation ignores the small second-order term ng, so that $b \approx n + g + \delta$.

Question 1: Equilibrium Dynamics, Steady State, and Balanced Growth [Total: 60 marks]

Consider the discrete-time Solow–Swan economy

$$Y_t = F(K_t, A_t L_t) = K_t^\alpha (A_t L_t)^{1-\alpha}, \qquad A_{t+1} = (1+g)A_t, \qquad L_{t+1} = (1+n)L_t,$$

$$K_{t+1} = (1-\delta)K_t + I_t, \qquad C_t = (1-s)Y_t, \qquad S_t = sY_t, \qquad I_t = S_t.$$

Define

$$k_t \equiv \frac{K_t}{A_t L_t}, \qquad y_t \equiv \frac{Y_t}{A_t L_t}, \qquad c_t \equiv \frac{C_t}{A_t L_t}.$$

(a) Derive the intensive-form production function and the law of motion

$$k_{t+1} = \mathcal{G}(k_t).$$

Then derive $k_{t+1} - k_t$ and explain the economic meaning of bk_t.

(b) Solve explicitly for the positive steady state k^*, y^*, and c^*. Then compute the exact and approximate steady-state values when

$$\alpha = \frac{1}{3}, \qquad s = 0.24, \qquad n = 0.01, \qquad g = 0.02, \qquad \delta = 0.05.$$

(c) Prove that the steady state is unique and globally stable for any $k_0 > 0$. Then derive the local convergence coefficient

$$\lambda \equiv \mathcal{G}'(k^*).$$

Compute λ and the half-life of deviations from steady state for the numerical parameters in part (b).

(d) Derive the balanced-growth-path growth rates of K_t, Y_t, C_t, I_t, K_t/L_t, and Y_t/L_t. Then derive the comparative statics of k^* and y^* with respect to s, n, g, and δ.

Question 2: Golden Rule, Dynamic Inefficiency, and Saving-Rate Policy [Total: 40 marks]

Continue with the Cobb–Douglas Solow economy from Question 1:

$$y = f(k) = k^\alpha, \qquad c^* = (1-s)f(k^*), \qquad sf(k^*) = bk^*.$$

(a) Define the Golden Rule level of capital as the level of k that maximises sustainable steady-state consumption:

$$c(k) = f(k) - bk.$$

Derive k_{GR}, y_{GR}, c_{GR}, and the saving rate s_{GR} that decentralises the Golden Rule steady state.

(b) Using

$$\alpha = \frac{1}{3}, \qquad s_0 = 0.24, \qquad n = 0.01, \qquad g = 0.02, \qquad \delta = 0.05,$$

compute k_{GR}, y_{GR}, c_{GR}, and s_{GR}. Compare the initial economy with $s_0 = 0.24$ to the Golden Rule allocation. Is the economy saving too much or too little?

(c) Suppose the government permanently raises the saving rate from $s_0 = 0.24$ to $s_1 = 0.40$. Compute the immediate effect on consumption per unit of effective labour, holding k fixed at the old steady state. Then compute the new steady-state values k_1^*, y_1^*, and c_1^*. Does the policy raise long-run consumption? Does it achieve the Golden Rule?

(d) Derive the local transition dynamics around the new steady state after the saving-rate change. Let

$$\widehat{k}_t \equiv \ln k_t - \ln k_1^*.$$

Show that locally $\widehat{k}_{t+1} \approx \lambda \widehat{k}_t$, and derive the approximate growth rate of output per worker during transition.